

Gerhard Höhne

Theory of relativity with aha-experiences

A vivid introduction
with thought experiments
for high school students,
college students,
teachers and...

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Gerhard Höhne, Im Hagen 5, 63768 Hösbach-Winzenhohl

Preface

As a student of physics it was difficult for me to accept the theory of relativity, because I got to know it in a very abstract form only. When I then had to teach it myself, one of my main concerns was to convey this theory in such a way that it would be convincing right from the start. To what extent I succeeded in this concern, the reader can judge on the basis of this brochure.

At the beginning, a thought experiment is carried out with the so-called experimental seesaw, on the basis of which the relations $\mathbf{m} = \frac{\mathbf{m}_0}{\sqrt{1-v^2/c^2}}$ and $\Delta E = \Delta \mathbf{m} \cdot c^2$ are derived in an easily understandable way. Such an introduction is met with much interest by the students, because the laws mentioned are known to them as significant.

The experimental seesaw mentioned above is a device that encourages students to perform experiments that lead very quickly to the sentence of center-of-mass and to theorem of momentum. I developed this device because, as a physics teacher in the upper school of a grammar school, I considered it useful to start mechanics with the momentum theorem, because the treatment of this theorem should precede the introduction of the force measure, contrary to the usual practice in school literature, since the definition of this quantity according to $F = d(\mathbf{m} \cdot \mathbf{v}) / dt$ only seems to make sense with reference to this law. Only with the momentum theorem can it be convincingly shown that $\Delta(\mathbf{m} \cdot \mathbf{v})$ is a suitable measure for an external action. I have written about the use of the experimental seesaw in the classroom repeatedly from 1987 to the year 2000 in the "Praxis der Physik", in "Physik in der Schule" and in the "MNU-Journal". Detailed reports on the use of the experimental seesaw in the classroom can be found on the website www.g-hoehne.de.

In my lessons, I was also interested in relating the theory of relativity to things that are well known to students. For this reason, I derived the laws of refraction of light at interfaces and interference of light behind a double slit using relativity how you can read it in Chapters 12 and 13. In 1982 I had published these derivations in a somewhat more complex form in the "Praxis der Physik" (PdN-Ph. 10/82).

I also published the contents of chapters 11 and 14, which appeared in 1992 in the journal "Physik in der Schule".

In usual school presentations of the theory of relativity I missed again and again a clarification of the twin paradox.

Therefore I dedicate a separate chapter to this paradox in this booklet, in which not only is clarified about this paradox, but furthermore arguments are given for the following theorem: **In a closed space, gravitational and inertial forces cannot be distinguished without knowledge of their causes. This theorem is the basis of the general theory of relativity.**

At the end of this paper you will find a page for comments. I would be very grateful if you would point out passages to me that are not linguistically correct. I would especially appreciate suggestions for improvement.

Gerhard Höhne

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1. The Principle of Relativity

In a uniformly moving railway carriage, a body falls as it would be in a room at rest according to Galileo's law of falling. It seems that such uniformity applies to every experiment. This means, there is no experiment in a closed, supposedly stationary space with which it can be decided whether this space is really at rest or whether it is moving uniformly instead. This statement is formulated as a principle of relativity as follows:

In spaces that move uniformly in relation to each other the same laws has to be applied.

In thought experiments with light signals, one has the impression that this principle of relativity leads to contradictions. Think of a spaceship flying at $v = 100000$ km/s from which a light signal is emitted in the direction of flight. This signal has the speed $c = 300000$ km/s from the point of view of an observer A at rest on earth, because the speed of light is not influenced by the speed of the light source. According to the principle of relativity, the signal should also have the speed $c' = 300000$ km/s from the point of view of a passenger B in the spaceship. This seems inappropriate to A, because from his point of view the spaceship follows the light signal at 100000 km/s. In his opinion, B should measure 200000 km/s as the speed of the light signal. However, the assertion $c' = 200000$ km/s is incompatible with the principle of relativity. The equation $c' = 300000$ km/s can be only correct if the measuring instruments in the spaceship behave differently due to fast movement than A imagines. **In such matters, experiments are desirable for clarification. In the present case, we must be content with thought experiments.**

The next chapter describes a thought experiment that is used to deduce the following sentence with respect of the principle of relativity: **The mass m as a measure of the inertia of an object is not constant, but increases according to $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$ with increasing speed v .** m_0 stands for the rest mass of the object and c for the speed of light. The equation given has proved to be correct in the calculation of the orbits of fast electrons in the magnetic field and thus stands for the validity of the principle of relativity.

2. The change of mass due to movement

In a spaceship moving away from its earthly launch pad at a speed of $v = 100000$ km/s a seesaw is set up (see Fig. 2.1) to compare the masses of two experimental waggons W_a and W_{a^*} , which have different speeds but are otherwise identical.

It is a very far to the left and right extending seesaw, which is attached at its middle M to a metal frame by two thin iron bands. At this bands the seesaw can rotate against the resistance of a leaf spring F (return spring) on its left edge.

F: Return spring

S: Sensor
for the electrical
measurement
of a small rotation

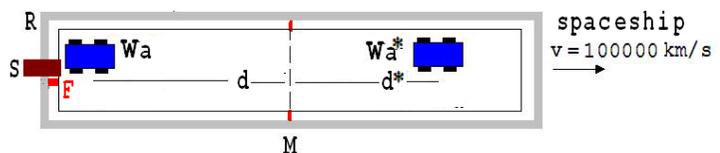


Fig. 2.1

A passenger B of the spaceship accelerates two identical experimental waggons W_a and W_{a^*} , which are initially at rest in the spaceship, completely uniformly to the left and to the right (in and against the direction of flight) in such a way that they cross the middle M of the seesaw at the same time and from then on glide over the seesaw uniformly without friction

from his point of view. It is assumed that the spaceship is already so far away from the Earth that the Earth's gravitational force has no appreciable significance for the processes in the spaceship and that, from B's point of view, the wagon W_a moves away from M just as quickly as the disappearing launch pad. W_a is accordingly at rest for an observer A standing on Earth. A can compare a wagon W_a at rest with a wagon W_{a^*} which, from his point of view, has the speed u .

In order to compare the masses of W_a and W_{a^*} , B disposes, that the frame of the seesaw in the spaceship is raised evenly along its entire length for a short time. Since W_a and W_{a^*} do not differ from his point of view and are equidistant from M due to the same velocities, the seesaw does not experience any rotation in its frame.

The observer A at rest cannot detect any rotation either. This means for him: The inertial forces F and F^* exerted by the wagons W_a and W_{a^*} on the seesaw satisfy the law of lever.

$$F \cdot d = F^* \cdot d^* \rightarrow a \cdot m \cdot d = a \cdot m^* \cdot d^* \rightarrow m \cdot d = m^* \cdot d^*$$

$F = a \cdot m$ and $F^* = a \cdot m^*$ are the inertial forces that W_a and W_{a^*} exert on the seesaw when it is lifted.

d and d^* : Distances of the wagons W_a and W_{a^*} from M as seen by A

m and m^* : Masses of the wagons W_a and W_{a^*} according to the point of view of A

a : Acceleration during lifting

$m \cdot d = m^* \cdot d^*$ can also be seen as a conclusion from the centre of gravity theorem. According to this theorem, the centre of gravity of the two wagons remains at the middle M of the seesaw from the point of view of both observers during the experiment described here. $m \cdot d = m^* \cdot d^*$ is a necessary condition for this.

If the experimenter B in the spaceship sends light signals to the wagons W_a and W_{a^*} from M at the same time, then, if the principle of relativity is valid, they must arrive at him again at the same time after reflection at the wagons. It is now shown that under this condition, from the point of view of observer A, the inequality $d^* < d$ applies and that the experimentally verifiable conclusion $m^* > m$ results from this.

The starting point for the investigation with the result $d^* < d$ is the following consideration:

If the running times to the mirror and back again of the light signals emitted to the right and left are the same for B, then this must also be the case for A. It is conceivable that the times valid for A can be determined with the help of d and d^* . In this case, the mathematical expressions with the variables d and d^* for the two times could be set equal to each other and we would have an equation that enables the comparison of d and d^* .

The following applies to the light signal directed at W_{a^*} :

Forward run: $c \cdot t_1 = d^* + u \cdot t_1 \rightarrow t_1 = \frac{d^*}{c - u}$

d^* : Distance of the wagon W_{a^*} from M at the start of the light signal

u : Speed of W_{a^*} in relation to A, t_1 : Time for the way in front, c : Speed of light

During the time t_1 , the path of the light to W_{a^*} will be elongated by $u \cdot t_1$. Thus the forward path of the light signal is not equal to d^* , but to $d^* + u \cdot t_1$.

Way back: $c \cdot t_2 = d^* + u \cdot t_1 - v \cdot (t_1 + t_2)$

t_2 : Return time of the light signal, v : Speed of the spaceship

During the running forward and back, M follows the light signal by $v \cdot (t_1 + t_2)$. Therefore, the way back is obtained by shortening the way out by $v \cdot (t_1 + t_2)$.

From the equations for the ways forward and back it follows:

$$c \cdot t_1 + c \cdot t_2 = 2 \cdot (d^* + u \cdot t_1) - v \cdot (t_1 + t_2) \rightarrow c \cdot (t_1 + t_2) + v \cdot (t_1 + t_2) = 2 \cdot (d^* + u \cdot t_1)$$

$$(t_1 + t_2) \cdot (c + v) = 2 \cdot (d^* + u \cdot t_1) \rightarrow t_1 + t_2 = 2 \cdot \frac{d^* + u \cdot t_1}{c + v}$$

In the right-hand part of the equation, the expression $\frac{d^*}{(c-u)}$ is inserted for t_1 .

$$\text{Running time } t_{\text{right}} = t_1 + t_2 = 2 \cdot \frac{d^* + \frac{u \cdot d^*}{c-u}}{c+v} = 2 \cdot \frac{d^* \cdot (c-u) + u \cdot d^*}{(c-u) \cdot (c+v)} = \frac{2 \cdot d^* \cdot c}{(c-u) \cdot (c+v)}$$

The following applies to the signal directed to the left on Wa:

Forward run: $c \cdot T_1 = d$, Way back: $c \cdot T_2 = d + v \cdot (T_1 + T_2)$

T_1 : Time for the way in front, T_2 : Return time, d : Distance of the wagon Wa from M at the start of the light signal

During the two ways forward and back, M moves away from Wa additional by $v \cdot (T_1 + T_2)$.

From the two equations for T_1 and T_2 it follows:

$$c \cdot (T_1 + T_2) = 2 \cdot d + v \cdot (T_1 + T_2) \rightarrow c \cdot (T_1 + T_2) - v \cdot (T_1 + T_2) = 2 \cdot d \rightarrow (T_1 + T_2) \cdot (c - v) = 2 \cdot d$$

$$\text{Running time } T_{\text{left}} = T_1 + T_2 = \frac{2 \cdot d}{c - v}$$

$$\text{Because of } T_{\text{left}} = t_{\text{right}} \text{ it applies: } \frac{2 \cdot d}{c - v} = \frac{2 \cdot d^* \cdot c}{(c - u) \cdot (c + v)} \rightarrow \frac{d^*}{d} = k = \frac{(c - u) \cdot (c + v)}{c \cdot (c - v)}$$

An equation is desired which shows the dependence of the ratio $d^*/d = k$ on the speed u of the wagon Wa^* . Therefore, the speed v of the spaceship must be replaced by a term dependent on u .

$$\text{It applies: } \frac{d + d^*}{d} = \frac{u}{v} \rightarrow 1 + \frac{d^*}{d} = \frac{u}{v} \rightarrow 1 + k = \frac{u}{v} \rightarrow v = \frac{u}{1+k}$$

Remark: $d + d^*$ (distance between Wa and Wa^*) and d are the distances, which Wa^* and M have covered from the point of view of observer A during a time t before the emission of the light signal.

$$(d^* + d) / t = u, \quad d / t = v \rightarrow (d + d^*) / d = u / v$$

$u / (1 + k)$ is substituted for v into the equation valid for $d^* / d = k$.

$$\frac{u}{1+k} \text{ is substituted for } v \text{ into the equation } k = \frac{(c-u) \cdot (c+v)}{c \cdot (c-v)} \rightarrow k = \frac{(c-u) \cdot (c + \frac{u}{1+k})}{c \cdot (c - \frac{u}{1+k})}$$

The last term is extended with $(1+k)$ and afterwards both sides are multiplied by $[c \cdot (c + c \cdot k - u)]$.

$$k = \frac{(c-u) \cdot (c + c \cdot k + u)}{c \cdot (c + c \cdot k - u)} \rightarrow \underline{k \cdot c^2} + k^2 \cdot c^2 - \underline{k \cdot c \cdot u} = c^2 + \underline{c^2 \cdot k} + \underline{c \cdot u} - \underline{u \cdot c} - \underline{u \cdot c \cdot k} - u^2$$

$$k^2 \cdot c^2 = c^2 - u^2 \rightarrow k^2 = 1 - \frac{u^2}{c^2} \rightarrow k = \frac{d^*}{d} = \sqrt{1 - u^2 / c^2}$$

Considering $d^* \cdot m^* = d \cdot m$ it follows: $\frac{d^*}{d} = \frac{m}{m^*}$, $\frac{m}{m^*} = \sqrt{1 - u^2/c^2} \rightarrow m^* = \frac{m}{\sqrt{1 - u^2/c^2}}$

m is the mass of the wagon W_a at rest from A 's point of view, which is also assigned to the wagon W_a^* at rest, because the two wagons are equal to each other.

$m = \text{rest mass} / \sqrt{1 - u^2/c^2}$ is thus the correct measure for the inertia of a body. For the rest mass ($u = 0$) we write m_0 from now on.

The diagram (Fig. 2.2) shows how the mass of a body ($m_0 = 1\text{kg}$) changes with increasing speed. Only at velocities in the order of magnitude of the speed of light clear deviations from the rest mass can be seen.

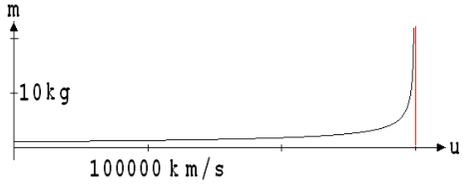


Fig. 2.2

3. Mass and Energy

The mass of a body increases with its kinetic energy. Using the thought experiment described in chapter 2 (Fig. 2.1), it is easy to derive a relationship between the increase in mass and the kinetic energy.

When the rear edge of the seesaw collides with the stationary wagon W_a of mass m , W_a is reflected elastically from B 's point of view. A , on the other hand, sees this process as an acceleration from rest to velocity u . W_a is somewhat compressed and consequently adheres for a time Δt to the left edge of the seesaw, being accelerated by this edge with a force F along a distance of length $v \cdot \Delta t$. The seesaw thus performs the work $W = F \cdot v \cdot \Delta t$ against W_a . Since the momentum of W_a is equal to 0 before the impact and equal to $(m_0/k) \cdot u$ afterwards, $(m_0/k) \cdot u$ stands for the change of momentum of the wagon during the time Δt .

$$F = \text{momentum change} / \text{time of change} = \frac{m \cdot u}{\Delta t} = \frac{m_0 \cdot u}{k \cdot \Delta t}, \quad m = \frac{m_0}{k}, \quad k = \sqrt{1 - u^2/c^2}$$

For critical readers: F is to be understood here as the time average value of the force over Δt .

Accordingly it applies for the work W performed against W_a :

$$W = F \cdot v \cdot \Delta t = \frac{m_0 \cdot u \cdot v}{k}$$

Considering the equation $v = \frac{u}{1+k}$ derived in chapter 2, we obtain: $W = \frac{m_0 \cdot u^2}{k \cdot (1+k)}$

Expanding the fraction term with $(1-k)$ leads to:

$$W = \frac{m_0 \cdot u^2 \cdot (1-k)}{k \cdot (1-k^2)}, \quad k^2 = 1 - u^2/c^2 \rightarrow W = \frac{m_0 \cdot u^2 \cdot (1-k)}{k \cdot u^2/c^2} = \left(\frac{m_0}{k} - m_0 \right) \cdot c^2$$

$$\frac{m_0}{k} - m_0 = m - m_0 = \text{mass increase } \Delta m \rightarrow W = \Delta m \cdot c^2$$

Therefore the wagon W_a with the velocity u has the kinetic energy $E_{\text{kin}} = \Delta m \cdot c^2$. Now the question arises whether an object also changes its mass by Δm according to $\Delta E = \Delta m \cdot c^2$ if its energy decrease or increase by ΔE does not stand for a kinetic energy. In the 11th chapter it is proved that this question must be answered in the affirmative. For example, one can calculate the energy released in the fission or fusion of atomic nuclei. A prerequisite for such a calculation is knowledge of the total masses of the initial and final products. Since the mass of a body becomes infinitely large if its speed tends to the speed of light, it

can never be accelerated to the speed of light, since an infinitely high energy would be necessary for this.

The speed of light is a limit speed.

With $E_{\text{kin}} = m_0 \cdot u^2 / 2$ the kinetic energy of a body of mass m_0 with velocity u was calculated up to now. According to the above calculation it applies: $E_{\text{kin}} = \Delta m \cdot c^2 = \frac{m_0 \cdot u^2}{k \cdot (1+k)}$.

If u is considerably smaller than c , then k can be set equal to 1 and therefore it applies $\Delta m \cdot c^2 \approx m_0 \cdot u^2 / 2$.

4. The time dilation

The equations about the mass change have been confirmed experimentally. Thus the principle of relativity can be considered as correct. It has been pointed out that, according to this principle, measuring rods and clocks must behave differently in motion than at rest.

In this context, it must be made clear that the determination a space is at rest is arbitrary in nature. If a space S is defined as at rest, then we call a space S' moved, if S' changes its distance to S.

We shall now consider the question:

How do measuring rods and clocks change, if they are brought from a space at rest to a space in motion?

We imagine two clocks U and U' in the wagons W_a and W_a^* during the experiment indicated in Fig. 2.1. These clocks are set to time 0 at the beginning of the experiment at M . According to the principle of relativity, both clocks show the same time from the point of view of the spacecraft passenger B , when they reach the ends of the seesaw, which are equidistant from M . If the resting clock U has distance L from M at the left end from A 's point of view and shows time t , then U' is only $d^* = d \cdot \sqrt{1 - u^2/c^2} < d$ away from M from A 's point of view and therefore shows a smaller time t' . Moving clocks therefore go slower for A than clocks at rest.

How does t' relate to t ?

If the clock U at rest at a distance of d from M shows the time t , then U' at a distance d^* from M shows the smaller time t' and later with the larger distance d the time t . The times t' and t behave to each other like the distances d^* and d .

$$t'/t = d^* / d, \quad d^* / d = \sqrt{1 - u^2/c^2} \rightarrow t' = t \cdot \sqrt{1 - u^2/c^2} \rightarrow t = t' / \sqrt{1 - u^2/c^2}$$

What follows from this?

According to the principle of relativity, an event with a development determined by the law of nature takes the same time in a laboratory at rest as in a laboratory in motion, if measured with the clock of the laboratory. Let t_0 be the duration of an event in a laboratory flying with the speed u . An observer in rest with his faster ticking clock assigns to this event the longer time $t = t_0 / \sqrt{1 - u^2/c^2}$. From his point of view, the event is time-dilated as a result of the motion.

This **time dilation** allows the very fast flying muons (elementary particles) produced in the upper atmosphere to reach the Earth's surface despite their short half-life of 1.523 μs .

Without this relativistic effect, even if they were moving at the speed of light, half of all muons would have already decayed after about 450 meters.

5. Length contraction

Let us imagine that the two wagons W_a and W_{a^*} of Fig. 2.1, from the point of view of the experimenter B working in the spaceship, are dragging two equal laths directly behind them. From the point of view of an observer A at rest, the lath tied to W_a is at rest, while the one tied to W_{a^*} is moving with velocity u . The ends of these slats meet in the middle M of the seesaw during the motion. The lengths L and L^* of the slats are therefore not equal from the point of view of A, but behave like the distances d and d^* of the wagons W_a and W_{a^*} from M.

$$d^*/d = L^*/L = \sqrt{1 - u^2/c^2} \rightarrow L^* = L \cdot \sqrt{1 - u^2/c^2}$$

We see: The length of an object extending in the direction of motion decreases with increasing speed from the point of view of an observer at rest (length contraction). This does not apply to a rod which lies transverse to the direction of motion. The same lengths L^* and L are assigned to it by the stationary and co-moving observer.

Proof: As indicated in Fig. 5.1, in this case the co-moving observer B measures the length L of the rod [EF] in the following way:

He sends a light signal from E to F. At F this signal is reflected by a mirror and returns to E again. B specifies the value $c \cdot t'$ as the path length L . t' is the half transit time of the light signal measured by B.

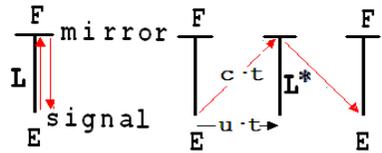


Fig. 5.1

Fig. 5.2

From the point of view of an observer A at rest, the light signal moves along the path shown in Fig. 5.2. While the light signal travels from E to F and back again, the rod [EF] moves to the right with the velocity u . A calculates the length L^* of [EF] according to the Pythagorean theorem: $c^2 \cdot t^2 - u^2 \cdot t^2 = L^{*2} \rightarrow L^* = c \cdot t \cdot \sqrt{1 - v^2/c^2}$

t is the half transit time of the light measured by A.

Because of $t = t' / \sqrt{1 - v^2/c^2}$ applies: $L^* = c \cdot t' = L$.

6. About the setting of clocks placed in different locations (synchronization).

Several clocks placed at different locations are to be set according to a clock Q. At first it seems advisable to proceed according to the following instruction: Take the clock Q, drive with it to the different places and set the clocks present there according to Q in each case. If one thinks about this procedure, it turns out to be deficient, since the rate of the clock Q depends on the travel speed. The following method is better: To set a clock R in a room S after another clock Q standing in S, a place P is determined in S, which lies exactly between R and Q. Flashes of light are emitted from this location. Here, the clock R is set so that R and Q show equal times when a light signal coming from P arrives at them. If the clocks are synchronized in the same way in a space S' moving with the velocity v , then it will be experienced that events which occur simultaneously from the point of view of a co-moving

observer B are not always simultaneous for an observer A at rest. This fact will be examined here in detail on the basis of fig. 6.1.

Let S be a space at rest, S' a space moving with velocity v. These spaces are assigned x,y coordinate systems (see Fig. 6.1). Let the two clocks Q and R in S' be set in such a way that a light signal emitted from a point P exactly in between reaches Q and R at the same time t' (for the same indication).

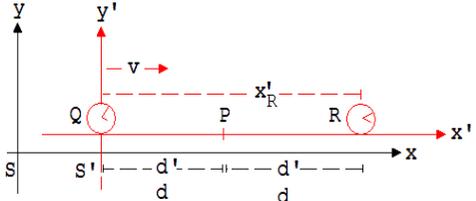


Fig. 6.1

d: distances P-Q and P-R with respect to the observer A at rest

x'_R , d': distances with respect to a co-moving observer B

Seen from A, the light signal takes the time t_1 to the clock Q and the somewhat longer time t_2 to the clock R.

$$t_1 \cdot c = d - v \cdot t_1 \rightarrow t_1 = \frac{d}{c+v}, \quad t_2 \cdot c = d + v \cdot t_2 \rightarrow t_2 = \frac{d}{c-v}$$

$$\Delta t = t_2 - t_1 = \frac{d}{c-v} - \frac{d}{c+v} = \frac{2 \cdot d \cdot v}{c^2 - v^2} = \frac{2 \cdot d \cdot v}{c^2 \cdot k^2}, \quad k^2 = 1 - v^2/c^2$$

Consequently, it reaches R later than Q with the time difference $\Delta t = t_2 - t_1$. Consequently, it reaches R later than Q with the time difference $\Delta t = t_2 - t_1$. From A's point of view, the pointer of clock Q is turned forward with respect to R. When the light signal reaches the clock Q, then this shows the time t' . Then the time Δt passes until the light signal reaches R and R then shows the time t' . During Δt , Q continues to run from the point of view of A by $\Delta t' = \Delta t \cdot k$. Afterwards Q and R show different times for A at the same time t, namely $t' + \Delta t'$ and t' . Considering the time dilation $\Delta t = \Delta t'/k$ and the length contraction $x'_R \cdot k = 2 \cdot d$ we find:

$$\frac{\Delta t'}{k} = \frac{x'_R \cdot k \cdot v}{c^2 \cdot k^2} \rightarrow \Delta t' = \frac{x'_R \cdot k^2 \cdot v}{c^2 \cdot k^2} = \frac{x'_R \cdot v}{c^2}$$

The display of Q is larger by $\Delta t'$ than that of R.

7. The Lorentztransformations

A space S' moves with velocity v with respect to a space S at rest. S and S' are provided with coordinate systems whose x-axes lie on each other and whose y- and z-axes are parallel (see Fig. 7.1). Let S' be equipped with clocks and length units which are also common in resting space. The clocks set up at the different locations of S' had been synchronized with each other according to the given rules and, like the clocks in S, had been set to 0 while the two y-axes touched each other.

At a point P an event is perceived by a moving observer B and an observer A at rest.

t is the time of the event measured by A and x is the x-coordinate of the point P measured by A.

t' and x' are the corresponding values of a observer B in S'.

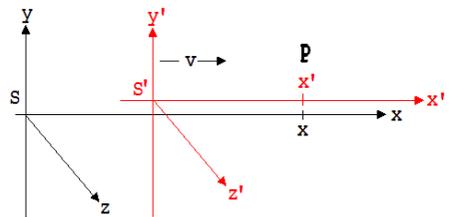


Fig. 7.1

How can the values belonging to one coordinate system be calculated if the values of the other system are known ?

$x - v \cdot t$ is the distance of the point P measured by A from the coordinate origin of the moved coordinate system. The observer B measures this distance with shortened units (length contraction!) and therefore assigns the larger value x' to it.

$$x' \cdot k = (x - v \cdot t) \quad \rightarrow \quad \mathbf{1.) \quad x' = (x - v \cdot t) / k, \quad k = \sqrt{1 - v^2/c^2}}$$

At time t , A reads the time t'_0 on the clock standing at the zero point of the x',y' system. Taking time dilation into account, the following applies: $t'_0 = t \cdot k$.

The moving clock goes slower !

The time t' , which A reads at the same time t on the clock located in S' at place P, is smaller than t'_0 by $x' \cdot v/c^2$.

$$t'_0 = t' + x' \cdot v/c^2, \quad t'_0 = t \cdot k \quad \rightarrow \quad t \cdot k = t' + x' \cdot v/c^2 \quad \rightarrow \quad \mathbf{2.) \quad t = (t' + x' \cdot v/c^2) / k}$$

Equations 1 and 2 also allow the calculation of x when x' and t' are known and that of t' when x and t are known. In the first case, t in the 1st equation must be replaced by the term $(t' + x' \cdot v/c^2) / k$ standing for t and the new equation thus obtained must be solved for x . In the second case x' of the 2nd equation must be replaced by the term $(x - v \cdot t) / k$.

After this, the new equation is to be dissolved for t' . Thus one receives altogether 4 equations known as Lorentztransformations. They are compiled in the following table.

$$\begin{array}{ll} x' = (x - v \cdot t) / k & t' = (t - x \cdot v/c^2) / k \\ x = (x' + v \cdot t') / k & t = (t' + x' \cdot v/c^2) / k \end{array}$$

Since the distances lying transverse to the direction of motion do not experience length contractions, the following applies to the y and z coordinates of a point lying off the x axis:

$$y = y' \text{ and } z = z' .$$

It is now to be checked whether the Lorentz transformations are in agreement with the principle of relativity.

The Lorentz transformations are acceptable only if with them the following conclusions from the relativity principle can be justified:

1. If an observer in S sees the clocks in S' going slower than a clock in S , then for an observer in S' the clocks in S must go slower than those in S' .
2. If the observer in S assigns a length $< 1\text{m}$ to a 1m scale lying in S' , then an observer in S' must also see a 1m scale lying in S shortened.
3. If a light wave in S propagates to all sides with the velocity c , then this must also be true in S' .

To 1.: Let a clock U be set up in S at a place with the coordinate x (see Fig. 7.2). When this clock indicates the time t_1 , the indicator of a clock U'_1 in S' , which is immediately adjacent to it, is at t'_1 . In the following time, U'_1 moves away from U , other clocks of the spaceship reach U . When U indicates the time t_2 , the clock U'_2 , which is then adjacent to it, indicates the time t'_2 . The time $t_2 - t_1$ measured by the clock U is assigned the time $t'_2 - t'_1$ by the observer B in S' .

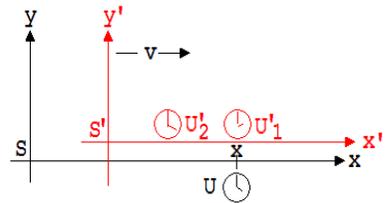


Fig. 7.2

$$\text{It applies: } t'_1 = (t_1 - v \cdot x / c^2) / k, \quad t'_2 = (t_2 - v \cdot x / c^2) / k$$

$$t'_2 - t'_1 = (t_2 - t_1) / k \rightarrow t'_2 - t'_1 > t_2 - t_1$$

From $t'_2 - t'_1 > t_2 - t_1$ B concludes: The clocks in S run slower than those in S'.

The resting observer A counters that t'_2 is too large because the clock U'_2 is ahead of the clock U'_1 . According to his opinion, $t'_2 - t'_1$ is too large.

We notice that the observers A and B come to completely contradictory appearing results when comparing their clocks. This is due to the fact that A, like B, compares a clock moving from his point of view with two clocks of his space which show different times seen from the other space.

To 2: Let $[x_1 \ x_2]$ be a distance on the x-axis of a coordinate system at rest. Observers in S' measure the x'-values belonging to x_1 and x_2 according to their clocks simultaneously at the time t' and assign the length $x'_2 - x'_1$ to the distance $[x_1 \ x_2]$. From the point of view of an observer A at rest, x'_2 is measured after x'_1 because from his point of view the clock at x'_2 is lagging in comparison with the clock at x'_1 .

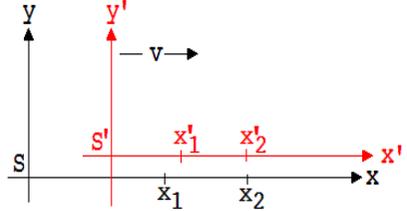


Fig. 7.3

According to his perception, the marker applied at x'_1 for the beginning of the distance $[x_1 \ x_2]$ moves to the right before the determination of x'_2 , which is why the difference $x'_2 - x'_1$ turns out to be too small in his opinion.

According to the Lorentz transformations it applies: $x_1 = (x'_1 + v \cdot t')/k$, $x_2 = (x'_2 + v \cdot t')/k$

$$x_2 - x_1 = (x'_2 - x'_1)/k \rightarrow x'_2 - x'_1 = (x_2 - x_1) \cdot k \rightarrow x'_2 - x'_1 < x_2 - x_1$$

It was shown earlier that moving distances parallel to the direction of propagation appear smaller to the observer at rest than equal distances at rest. Since this seems to contradict the inequality just derived, it must be pointed out that observers at rest measure the coordinates x_1 and x_2 of the end points of a distance $[x'_1 \ x'_2]$ according to clocks at rest simultaneously, while moving observers determine the coordinates x'_1 and x'_2 to the end points of a distance $[x_1 \ x_2]$ according to the clocks of their space simultaneously.

To 3.: At the time $t = t' = 0$, a spherical light wave is emitted at the common coordinate origin of S and S'. Let P be a point which lies on the wave front of the spherical wave at a time (t, t') . For its distance L from the coordinate origin of the system S applies:

$$L^2 = x^2 + y^2 + z^2 = c^2 \cdot t^2$$

From this it follows, taking into account the Lorentz transformations:

$$(x' + v \cdot t')^2 / k^2 + y'^2 + z'^2 = c^2 \cdot (t' + v \cdot x' / c^2)^2 / k^2$$

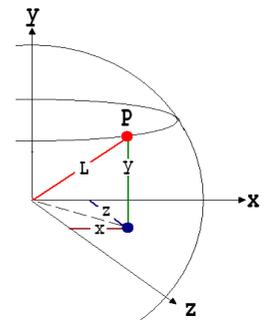


Fig. 7.4

$$x'^2 + v^2 \cdot t'^2 + 2 \cdot x' \cdot v \cdot t' + y'^2 \cdot k^2 + z'^2 \cdot k^2 = t'^2 \cdot c^2 + x'^2 \cdot v^2 / c^2 + 2 \cdot x' \cdot v \cdot t'$$

$$x'^2 - x'^2 \cdot v^2 / c^2 + y'^2 \cdot k^2 + z'^2 \cdot k^2 = t'^2 \cdot c^2 - v^2 \cdot t'^2$$

$$x'^2 \cdot (1 - v^2 / c^2) + y'^2 \cdot k^2 + z'^2 \cdot k^2 = t'^2 \cdot c^2 \cdot (1 - v^2 / c^2); \quad k^2 = (1 - v^2 / c^2)!$$

$$x'^2 + y'^2 + z'^2 = c^2 \cdot t'^2$$

According to this, the wave in S' moves away from the origin of coordinates as in S to all sides with velocity $\sqrt{(x'^2 + y'^2 + z'^2)} / t' = c$. We see that to set up the Lorentz transformations, both S and S' can be considered to be at rest. In both spaces the same conditions are given with respect to the other space.

8. The addition theorem of velocities

A space S' moves with the velocity v in relation to a space S defined as rest. An object P flies with the velocity u' in S' at the time $t' = 0$ through the zero point of the coordinate system belonging to S' . At the time $t = t' = 0$ the coordinate system of S' coincides with the coordinate system of S . The velocity vector u has the velocity coordinates u_x and u_y in the system S and the coordinates u'_x and u'_y in S' .

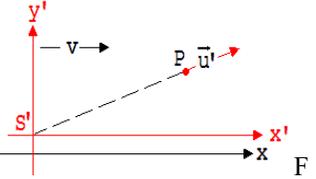


Fig. 8.1

How can u_x and u_y be calculated knowing u'_x and u'_y ?

$$u_x = x / t, \quad x = (x' + v \cdot t') / k, \quad t = (t' + v \cdot x' / c^2) / k, \quad k = \sqrt{1 - v^2 / c^2}$$

$$u_x = \frac{x}{t} = \frac{x' + v \cdot t'}{t' + v \cdot x' / c^2} = \frac{x' / t' + v}{1 + v \cdot (x' / t') / c^2}, \quad \frac{x'}{t'} = u'_x \rightarrow u_x = \frac{u'_x + v}{1 + v \cdot u'_x / c^2}$$

$$u_y = y / t, \quad y = y', \quad y' / t' = u'_y, \quad t = (t' + v \cdot x' / c^2) / k$$

$$u_y = \frac{y' \cdot k}{t' + v \cdot x' / c^2} = \frac{(y' / t') \cdot k}{1 + v \cdot (x' / t') / c^2} = \frac{u'_y \cdot k}{1 + v \cdot u'_x / c^2}$$

In a corresponding way, one obtains for u'_x and u'_y : $u'_x = \frac{u_x - v}{1 - v \cdot u_x / c^2}$, $u'_y = \frac{u_y \cdot k}{1 - v \cdot u_x / c^2}$

Propagation of light in flowing water:

After derivation of a law, an example of application is desirable. If such an example leads to a provable result, then one has a proof for the correctness of the law.

With the derived addition law, the speed with which light propagates in flowing water in the direction of flow can be calculated (see Fig. 8.2).

The flow velocity of the water is v . The light has the velocity c_w with respect to an observer floating with the water flow.

The ratio c/c_w ($c = c_{\text{vacuum}}$) is called the refractive index n of water.



Fig. 8.2

$$c_w = c/n$$

According to the addition law of velocities, we find for the speed of light c_{FW} in flowing water with respect to an observer at rest:

$$c_{FW} = \frac{c/n + v}{1 + (v \cdot c/n) / c^2} = \frac{c/n + v}{1 + v / (c \cdot n)}$$

For the change in velocity of light caused by the flow velocity v , we obtain:

$$\Delta c = c_{FW} - c/n = \frac{c/n + v}{1 + v / (c \cdot n)} - c/n = \frac{c/n + v - c/n \cdot [1 + v / (c \cdot n)]}{1 + v / (c \cdot n)} = \frac{v - v/n^2}{1 + v / (c \cdot n)} = v \cdot \frac{1 - 1/n^2}{1 + v / (c \cdot n)}$$

Considering $v \ll c$ (v is negligibly small compared to c), we can also write: $\Delta c = v \cdot (1 - 1/n^2)$. The last equation was established by the French physicist Fizeau in the 19th century on the basis of measured data.

9. Transformation of mass and momentum

Momentums and masses are transformed like location coordinates and times.

$$\begin{aligned} \mathbf{p}'_x &= (\mathbf{p}_x - \mathbf{m} \cdot \mathbf{v}) / k & \mathbf{m}' &= (\mathbf{m} - \mathbf{p}_x \cdot \mathbf{v} / c^2) / k & \mathbf{p}'_y &= \mathbf{p}_y \\ \mathbf{p}_x &= (\mathbf{p}'_x + \mathbf{m}' \cdot \mathbf{v}) / k & \mathbf{m} &= (\mathbf{m}' + \mathbf{p}'_x \cdot \mathbf{v} / c^2) / k & \mathbf{p}'_z &= \mathbf{p}_z \end{aligned}$$

The proof of these equations derived by Max Planck shall only be indicated here. It is started with the derivation of the equations in the 2nd column. Since the rest masses $m_0 = m' \cdot \sqrt{1 - u'^2/c^2}$ and $m_0 = m \cdot \sqrt{1 - u^2/c^2}$ are equal in the spaces S and S', the following equation with m and m' can be set up immediately. u and u' are the velocities in S and S'.

$$m \cdot \sqrt{1 - u^2/c^2} = m' \cdot \sqrt{1 - u'^2/c^2} \leftrightarrow m \cdot \sqrt{1 - (u_x^2 + u_y^2)/c^2} = m' \cdot \sqrt{1 - (u_x'^2 + u_y'^2)/c^2}$$

The y-axis is placed so that there is no movement in the z-direction. The velocity coordinates u_x and u_y are replaced by u'_x and u'_y according to the addition theorem. After an extensive algebraic transformation one finds:

$$\sqrt{1 - (u_x^2 + u_y^2)/c^2} = \frac{k \cdot \sqrt{1 - u'^2/c^2}}{1 + v \cdot u'_x/c^2}, \quad k = \sqrt{1 - v^2/c^2} \rightarrow m \cdot \frac{k \cdot \sqrt{1 - u'^2/c^2}}{1 + v \cdot u'_x/c^2} = m' \cdot \sqrt{1 - u'^2/c^2}$$

$$m = m' \cdot (1 + v \cdot u'_x/c^2) / k, \quad m' \cdot u'_x = p'_x \rightarrow \mathbf{1.} \quad \mathbf{m} = (\mathbf{m}' + \mathbf{p}'_x \cdot \mathbf{v} / c^2) / k$$

For m' one obtains in a similar way: **2.)** $\mathbf{m}' = (\mathbf{m} - \mathbf{p}_x \cdot \mathbf{v} / c^2) / k$

From the equations for m and m' the transformation equations for p_x and p'_x can easily be derived. $p'_x = (\mathbf{p}_x - \mathbf{m} \cdot \mathbf{v}) / k$ is obtained, for example, by replacing m' in the 1st equation by $(\mathbf{m} - \mathbf{p}_x \cdot \mathbf{v} / c^2) / k$ and then solving the new equation to p'_x .

Proof of the equation $\mathbf{p}_y = \mathbf{p}'_y \leftrightarrow \mathbf{m} \cdot \mathbf{u}_y = \mathbf{m}' \cdot \mathbf{u}'_y$

m' is replaced by $(\mathbf{m} - \mathbf{p}_x \cdot \mathbf{v} / c^2) / k$ and u'_y by $u_y \cdot k / (1 - v \cdot u_x / c^2)$ according to the addition theorem of velocities.

$$m' \cdot u'_y = \frac{(\mathbf{m} - \mathbf{p}_x \cdot \mathbf{v} / c^2)}{k} \cdot \frac{u_y \cdot k}{1 - v \cdot u_x / c^2}, \quad p_x = m' \cdot u_x \rightarrow \mathbf{p}'_y = \mathbf{p}_y$$

10. Transformation of the force

1. Transformation of a force in the direction of the x'-axis

Let us imagine a body K accelerated by a force F'_x .

How big is F_x ?

$$F_x = \frac{\Delta p_x}{\Delta t} = \frac{(\Delta p'_x + v \cdot \Delta m') / k}{(\Delta t' + \Delta x' \cdot v / c^2) / k} = \frac{\Delta p'_x / \Delta t' + v \cdot \Delta m' / \Delta t'}{1 + (\Delta x' / \Delta t') \cdot (v / c^2)}$$

$\Delta m' \cdot c^2 = F'_x \cdot \Delta x'$ is the work done in S' against K.

$$\Delta m' \cdot c^2 = F'_x \cdot \Delta x' \rightarrow \Delta m' = F'_x \cdot \Delta x' / c^2$$

$$\Delta p'_x / \Delta t' = F'_x !$$

$$F_x = \frac{F'_x + v \cdot (F'_x \cdot \Delta x' / c^2) / \Delta t'}{1 + (\Delta x' / \Delta t') \cdot v / c^2} = F'_x \cdot \frac{1 + (\Delta x' / \Delta t') \cdot (v / c^2)}{1 + (\Delta x' / \Delta t') \cdot (v / c^2)} \rightarrow \mathbf{F_x = F'_x}$$

$\Delta m' \cdot c^2 = F'_x \cdot \Delta x'$ and thus also $F_x = F'_x$ are valid only, if F'_y and F'_z are equal to 0.

2. Transformation of a force in the direction of the y-axes

$F_y = \Delta p_y / \Delta t$ und $F'_y = \Delta p'_y / \Delta t'$ are the forces parallel to the y-axes in S and S'.

$$F_y = \frac{\Delta p_y}{\Delta t}, \Delta p_y = \Delta p'_y, \Delta t = \frac{\Delta t' + \Delta x' \cdot v / c^2}{k} \rightarrow F_y = \frac{\Delta p'_y \cdot k}{\Delta t' + \Delta x' \cdot v / c^2} = \frac{(\Delta p'_y / \Delta t') \cdot k}{1 + (\Delta x' / \Delta t') \cdot v / c^2}$$

$$\downarrow$$

$$\frac{\Delta p'_y}{\Delta t'} = F'_y, \frac{\Delta x'}{\Delta t'} = u'_x, F_y = \frac{F'_y \cdot k}{1 + u'_x \cdot v / c^2}$$

In the case $u'_x = 0$ applies: $F_y = F'_y \cdot k$

11. Theorem of the conservation of mass

We imagine S' as a closed system of several particles 1, 2, ... with velocity v. The x-coordinate of the momentum of the i. particle we call p_i . To the sum Σp_i of the momentum coordinates p_i of all particles applies:

$$\Sigma p_i = (\Sigma p'_i + v \cdot \Sigma m'_i) / k \rightarrow \Sigma m'_i = (k \cdot \Sigma p_i - \Sigma p'_i) / v.$$

From the last equation, considering the momentum theorem (Σp_i is constant in a closed system), we can draw the following conclusion:

The total mass of a closed system is constant.

The equation $E = \Delta m \cdot c^2$ is therefore not only valid for the kinetic energy, but for every kind of energy. For example, if a flying body is decelerated by a spring, the total mass of the flying object and the spring remains constant. While the mass of one object increases, that of the other decreases. The law of conservation of mass corresponds to the law of conservation of energy. The mass increase of a spiral spring during compression can also be calculated using the thought experiment shown in Fig. 10.1. In a space S' moving uniformly with Velocity v, a spiral spring of mass m is compressed by d' with the left end of the spring held at a location with x' -coordinate 0 (see Fig. 10.1). The observer B calculates the work done in S' according to $F' \cdot d'$ under the assumption that d' is very small and thus the force F' can be considered constant during the deformation. $F' \cdot d'$ is for B the energy increase $\Delta E'$ of the

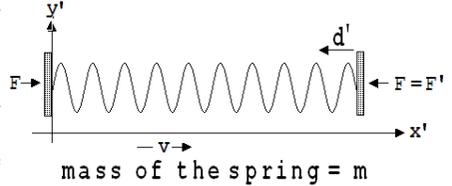


Fig. 10.1

spring during compression. From his point of view, the spring does not receive any momentum because the force F acting from the right causes an equally large counterforce against the left end of the spring (the right end of the spring is displaced uniformly). A resting observer A sees it differently. For him, the forces acting from the left and right are also of equal magnitude, but not their effective times Δt_L and Δt_R . From A's point of view, the compressive forces of size F act at the left end from time t_{1L} to time t_{2L} and at the right end from t_{1R} to t_{2R} . For B the compression at both the left and right ends by d' happen between times t'_1 and t'_2 .

$$t_{1L} = t'_1 / k, \quad t_{2L} = t'_2 / k, \quad t_{1R} = [t'_1 + (v/c^2) \cdot x'_1] / k, \quad t_{2R} = [t'_2 + (v/c^2) \cdot x'_2] / k$$

x'_1, x'_2 : x' values from the right end of the spring before and after compression

$\Delta t_L = t_{2L} - t_{1L}$, $\Delta t_R = t_{2R} - t_{1R}$: The effective times of the two forces from A's point of view.

$$\Delta t_L = (t'_2 - t'_1) / k, \quad \Delta t_R = t_{2R} - t_{1R} = [(t'_2 - t'_1) + (v/c^2) \cdot (x'_2 - x'_1)] / k$$

$$\Delta t_R = t_{2R} - t_{1R} = [(t'_2 - t'_1) - (v/c^2) \cdot d'] / k, \quad d' = x'_1 - x'_2$$

For the observer A, the force on the left end of the spring resting in S' thus acts during a time interval $\Delta t = \Delta t_L - \Delta t_R$ without counterforce and accordingly causes a change in momentum of the spring according to $\Delta(m \cdot v) / \Delta t = F \rightarrow \Delta(m \cdot v) = F \cdot \Delta t$.

Since the velocity v of the spring does not change, the following applies: $\Delta(m \cdot v) = v \cdot \Delta m$.

$$v \cdot \Delta m = F \cdot \Delta t = F \cdot (\Delta t_L - \Delta t_R) = F \cdot (v/c^2) \cdot d' / k, \quad F = F' \rightarrow \Delta m \cdot k = (1/c^2) \cdot d' \cdot F'$$

$\Delta m \cdot k$ is the change of the mass at rest $\Delta m'$ in S'.

$$\text{Thus applies: } \Delta m' = (1/c^2) \cdot d' \cdot F' \rightarrow \Delta m' \cdot c^2 = d' \cdot F' = \Delta E'$$

12. The law of refraction of optics

If a light ray falls on a plane interface between transparent materials, it is almost always refracted. The ratio of the sine of the angle of incidence α and the sine of the angle of refraction β is constant. $\sin \alpha / \sin \beta$ is called the refractive index n . n depends on the materials forming the interface. This law of refraction can be derived with the help of the theory of relativity. For this purpose, in a space S at rest, we imagine a cuboid of glass to which a light beam is directed (see Fig. 12.1).

Let the plane wave front of the light represented by the red line pass at the time $t = 0$ with the velocity c_L (velocity of light in air) through the coordinate origin of the x, y coordinate system belonging to S. Let P be a point on this wave front. For its y -value holds:

$$y = c_L \cdot t / \cos \alpha - x \cdot \tan \alpha.$$

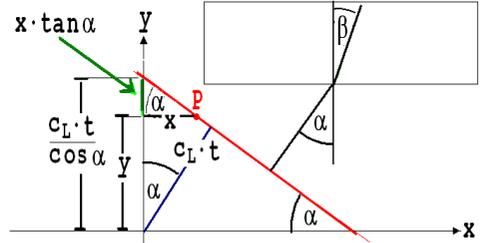


Fig. 12.1

We now think of another space S' moving in the direction of the x -axis with velocity v with an x', y' coordinate system whose axes coincide with those of the x, y system at the time $t = t' = 0$.

v can be chosen in such a way that the wavefront is parallel to the x' -axis from the point of view of an observer B in S'. This is the case if the y' belonging to a point P of the wave front is independent of x' .

$$y = c_L \cdot t / \cos \alpha - x \cdot \tan \alpha, \quad t = (t' + x' \cdot v/c^2) / k, \quad x = (x' + v \cdot t') / k$$

$$y = y' = (c_L / \cos \alpha) \cdot (t' + x' \cdot v/c^2) / k - [(x' + v \cdot t') / k] \cdot \tan \alpha$$

$$y' = (c_L / \cos \alpha - v \cdot \tan \alpha) \cdot t' / k + [(c_L / \cos \alpha) \cdot v/c^2 - \tan \alpha] \cdot x' / k$$

In the case $(c_L / \cos \alpha) \cdot v/c^2 - \tan \alpha = 0$, the wavefront in S' is parallel to the x' -axis.

$$(c_L / \cos \alpha) \cdot v/c^2 - \tan \alpha = 0 \rightarrow (c_L / \cos \alpha) \cdot v/c^2 - \sin \alpha / \cos \alpha = 0 \rightarrow \sin \alpha = v \cdot c_L / c^2$$

Since in S' the wave front remains parallel to the x' -axis even after entering the glass, a corresponding equation is also valid for the refracted wave in the glass.

$$\sin \beta = v \cdot c_G / c^2, \quad c_G = \text{Speed of light in a glass.}$$

$$\sin \alpha = v \cdot c_L / c^2, \quad \sin \beta = v \cdot c_G / c^2 \rightarrow \sin \alpha / \sin \beta = c_L / c_G = n$$

13. The light interference behind a double slit

In a space S at rest, a light beam is directed onto a double slit with the grating constant g (see Fig. 13.1). The beam is parallel to the y-axis of the coordinate system belonging to S. g stands for the distance between the two slits, which are assigned the values $-g/2$ and $+g/2$ on the x-axis. On a screen above are intensity maxima and intensity minima of light.

To understand this phenomenon, we imagine a space flying with velocity $v > 0$ in x-direction with an x', y' - coordinate system, whose x' -axis runs directly above the screen and whose y' -axis crosses the y -axis of the other system at the time $t = t' = 0$. v is chosen in such a way that the light oscillations (oscillation time = T) in the two slits have the same phases from the point of view of an observer B in S' . Wave crests appearing simultaneously at a time t' in the left and right slits from the point of view of B are seen from the point of view of an observer A in S as wave crests hitting the double slit at different times t_1 and t_2 with the temporal distance $n \cdot T$ ($n = 1; 2; 3, \dots$). $n = 0$ is to be excluded because the wave crests in the two slits cannot be seen simultaneously by A and B in the case $v > 0$.

$$\begin{aligned}
 t' &= [t_1 - (v/c^2) \cdot (-g/2)] / k \\
 t' &= [t_2 - (v/c^2) \cdot (g/2)] / k \\
 t_1 - (v/c^2) \cdot (-g/2) &= t_2 - (v/c^2) \cdot (g/2) \\
 t_2 - t_1 &= (v/c^2) \cdot g = n \cdot T \\
 c = \lambda / T &\rightarrow T = \lambda / c, \lambda : \text{Wavelength} \\
 n \cdot T &= n \cdot (\lambda / c) = (v/c^2) \cdot g \\
 &\downarrow \\
 n \cdot \lambda &= g \cdot v/c
 \end{aligned}$$

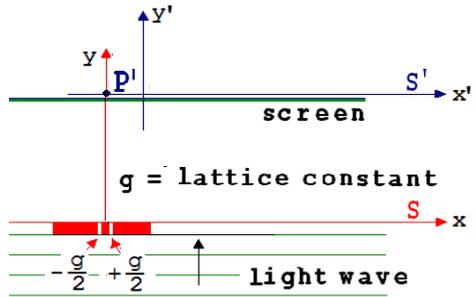


Fig. 13.1

From the point of view of an observer B in S' , two wave crests, which leave the slits at the time t' , run towards a point P' in S' lying exactly above the double slit. If the crests arrive at P' , then P' from the point of view of A has moved within time Δt to the right to a location on the screen that is $x = v \cdot \Delta t$ away from the y -axis. The wave valleys and crests following the crests mentioned above, which from the point of view of B simultaneously leave the slits and run parallel to the y' axis towards the x' axis, also arrive there. Therefore A sees an intensity maximum at this place.

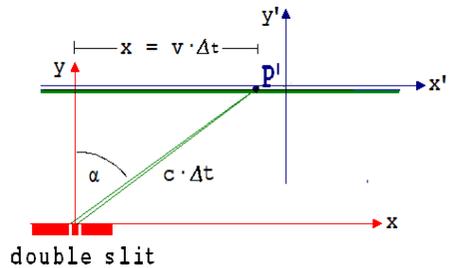


Fig. 13.2

$$n \cdot \lambda = g \cdot v/c, v/c = v \cdot t / (c \cdot t) = \sin \alpha \rightarrow n \cdot \lambda = g \cdot \sin \alpha$$

For the 1st order maximum ($n = 1$!) applies: $\lambda = g \cdot \sin \alpha$

If S' is at rest in S ($v = 0$), then in both systems the maximum 0th order ($n = 0$) is perceived on the y -axis ($\sin \alpha = 0$).

Not only to the right of the y -axis, but also to the left of it such maxima arise. To substantiate this claim, the thought experiment explained here must be modified. One does not let S' fly to the right but to the left instead.

A minimum of intensity is at P' when seen from B a wave crest in one slit appears simultaneously with a wave valley in the other slit. From A's point of view, the times of these two events differ by $t = (2 \cdot n - 1) \cdot T/2$, $n = 1, 2, 3, \dots$

t can be $T/2$ or $3 \cdot T/2$ or $5 \cdot T/2$

Accordingly, the following applies to minimum of the intensity:

$$(2 \cdot n - 1) \cdot T/2 = (v/c^2) \cdot g \rightarrow (2 \cdot n - 1) / 2 \cdot \lambda = g \cdot \sin \alpha$$

Fig. 13.3 shows how two circular waves originating from the white marked slits overlap and form intensity maxima and minima (dark stripes) on the screen. The wave crests are shown in red and the wave troughs in green. a is the distance between the maxima order one and order zero.

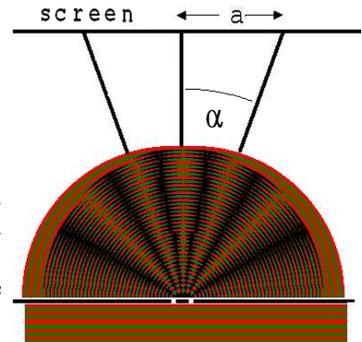


Fig. 13.3

14. $E = h \cdot f$

Derivation of Planck's relationship by means of a thought experiment

We imagine, as indicated in the sketch on the left, a mirror moving uniformly to the left with the velocity v , on which light of the frequency f falls. As is well known, the light has energy and therefore also a mass, accordingly it exerts a force on the moving mirror and performs a work which leads to a loss of energy of the light.

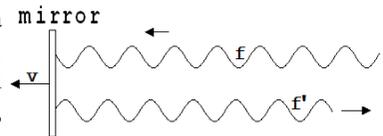


Fig. 14.1

In addition to the energy, the frequency also decreases. This is easy to see if one considers that the reflected signal and thus also the wavelength is stretched by the movement of the mirror. With the larger wavelength, a smaller frequency f' sets in after $\lambda - f = c$. Mostly energy losses of the light are not connected with a frequency change. Think e.g. of the light attenuation by a gray filter or by clouds.

Why is the energy loss accompanied by a frequency change in one case, but not in the other ?

To answer this question, one should look for similar phenomena. Imagine a gas in a vertical cylinder under a moving piston loaded with weights. If the weights are removed, then the gas expands, with the gas molecules doing work against the piston and consequently losing kinetic energy. Consequently, the temperature of the gas decreases. However, it is also possible to reduce the energy of the gas by allowing part of it to escape from its container. While in the first case the number of gas atoms remains constant and the energy of the individual atoms decreases, in the second case it is exactly the opposite. The number of atoms decreases, but the energy of individual atoms remains constant, which is why the temperature does not change. Under the impression of the mentioned facts now the following explanation to the mirror experiment is suggested: The light energy is divided into units of action, which we call photons. The moving mirror decreases the energy of the single photons, a filter swallows photons, and thus decreases the energy of a light beam. The frequency of the light depends on the energy of the photons.

What is the relationship between frequency and photon energy ?

The frequency and energy change during light reflection are now calculated.

1. Frequency change

A wave train of the length $n \cdot \lambda$ is stretched to the length $n \cdot \lambda'$ during its reflection of the duration t . If the beginning of the wave train has reached the mirror, the end has to travel the distance $n \cdot \lambda + v \cdot t$ in a time t until it arrives at the mirror.

$$n \cdot \lambda + v \cdot t = c \cdot t \rightarrow t = n \cdot \lambda / (c-v)$$

The beginning of the reflected wave train moves away during this time t from the mirror by

$$c \cdot t + v \cdot t = n \cdot \lambda' \rightarrow (c+v) \cdot t = n \cdot \lambda'$$

$$(c+v) \cdot t = n \cdot \lambda', \quad t = n \cdot \lambda / (c-v) \rightarrow (c+v) \cdot n \cdot \lambda / (c-v) = n \cdot \lambda'$$

$$(1.) \lambda / \lambda' = f' / f = (c-v) / (c+v)$$

2. Energy change

The mirror experiences the force $F = [m \cdot c - (-m' \cdot c)] / t$ in the reflection time t . m and m' are the light masses before and after the reflection, $m \cdot c$ and $m' \cdot (-c)$ are the momentums of the light before and after the reflection.

$$F = [m \cdot c - (-m' \cdot c)] / t = c \cdot (m + m') / t$$

To the work of light applies: $W = F \cdot v \cdot t = [c \cdot (m + m') / t] \cdot v \cdot t = v \cdot c \cdot (m + m')$

$v \cdot t$ describes the displacement of the mirror in time t .

To the work of the light is also true: $W = (m - m') \cdot c^2$

Thus we can write: $v \cdot c \cdot (m + m') = (m - m') \cdot c^2$

$$v \cdot (m + m') = c \cdot (m - m') \rightarrow m' \cdot (c + v) = m \cdot (c - v)$$

$$(2.) m' / m = (c-v) / (c+v)$$

From equations (1.) and (2.) follows: $f' / f = m' / m$

According to $E = m \cdot c^2$ the masses m and m' behave to each other like the associated energies E and E' .

$$f' / f = E' / E \rightarrow E' / f' = E / f$$

For the energies E_p and E'_p of the photons then also applies: $E'_p / f' = E_p / f = \text{constant}$.

This constant is called the quantum of action h .

$$E_p = h \cdot f$$

This law can be used to explain the photoelectric effect, on the basis of which the value $6.6261710 \cdot 10^{-34} \text{ J} \cdot \text{s}$ can be obtained for h .

15. The twin paradox

Imagine twins A and B on earth. B gets into a rocket and flies with $2/3$ speed of light through the space. After one year he comes back home to his brother. Since from the point of view of A all processes in the rocket are temporally stretched, B is then less aged than A. One could now give the following explanation known as twin paradox: A flies with the earth from the view of B with $2/3$ speed of light. What is valid for B must also be valid for A. With it one comes to the nonsensical conclusion: Each of the two twins is biologically younger than his twin brother.

The statement „What is valid for B must also be valid for A“ is not true here, because for B

the clocks on the earth go slower than the rocket clocks only when he has synchronized the clocks in his rocket after the acceleration process. At first, after the launch of his rocket, B sees the rocket clocks in a slower pace than the terrestrial clocks, just as his brother A does. The reason for this is the fact that the clocks in the rocket remain synchronous during launch from A's point of view and are therefore not synchronous from B's point of view. We assume that there are clocks in the rocket (see Fig. 15.1) which were set to 0 together with the terrestrial clocks immediately before the acceleration. During a uniform acceleration and afterwards, these clocks show coinciding times from the point of view of an observer A at rest.

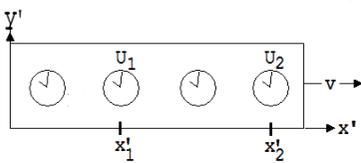


Fig. 15.1

From B's point of view, after acceleration to velocity v , clock U_2 at x'_2 is ahead of clock U_1 at x'_1 by $\Delta t' = (v/c^2) \cdot \Delta x'$. $\Delta x' = x'_2 - x'_1$!

Reasoning: If B has synchronized the clocks in the rocket after the clock U_1 , then the resting observer A determines, the clock U_2 is behind in comparison with U_1 by $(v/c^2) \cdot \Delta x'$. Therefore, it must have advanced by $(v/c^2) \cdot \Delta x'$ before synchronization for the co-moving observer B. Accordingly, the following applies to B directly after the acceleration process before the synchronization:

$$t'_2 = t'_1 + (v/c^2) \cdot \Delta x', \quad t'_2: \text{Time shown by the clock } U_2, \quad t'_1: \text{Time shown by the clock } U_1$$

For B the following explanation is conceivable: During the acceleration process with the acceleration $a' = (v/t'_1)$, the clock U_2 , under the effect of the inertial forces, assumes a faster rate than U_1 by $t'_2 / t'_1 = 1 + [(v/t'_1) / c^2] \cdot \Delta x' = 1 + [a' / c^2] \cdot \Delta x'$.

For a' he could also choose (v/t'_2) instead of (v/t'_1) . The difference between (v/t'_1) and (v/t'_2) is inconsequential for a rocket velocity below 1000 km/s. It can be seen that $v \ll c$ is assumed in the calculation of the acceleration.

B could also determine the acceleration of his rocket in the rocket by measuring in it the inertial force F_T acting on a body of mass m and then determining a' according to $a' = F_T / m$. Since inertial forces behave similarly to weight forces, it is reasonable to assume that weight forces could affect clocks in the same way as inertial forces.

The equation $t'_2 / t'_1 = 1 + [a' / c^2] \cdot \Delta x'$ should then also be valid for terrestrial clocks U_2 and U_1 at different altitudes, if one enters the difference in altitude h between a clock U_2 and a lower clock U_1 for $\Delta x'$ and replaces a' with $g = F_G / m$. F_G is the weight force on a body of mass m .

$$t_2 / t_1 = 1 + (g/c^2) \cdot h$$

t_2 and t_1 stand for the times which U_2 and U_1 indicate for the duration of a certain event. The rate difference expressed in the equation could be demonstrated with very accurate atomic clocks. At an altitude difference of 1000 m, the higher clock runs $4 \cdot 10^{-10}$ seconds ahead of the lower clock during one hour.

From this follows that a light wave experiences a decrease of its frequency (red shift) when it rises from a gravitational field. At higher places the clocks run faster. A larger oscillation time and thus a smaller frequency is measured there. Also this frequency change could be proved.

These facts lead to the conclusion:

In an enclosed space, gravitational and inertial forces cannot be distinguished without

knowledge of their causes.

This theorem is the basis of general relativity.

16. About history of the development of the theory of relativity

The special theory of relativity has many fathers. The General Theory of Relativity is the work of Einstein.

Notes on special relativity in the textbook for theoretical physics by Georg Joos (10th edition, 1959):

The time transformation has already made 1887 Voigt in a little known work. In 1892 Lorentz took it up for the optics of moving bodies. In 1900 Lamor wrote the Lorentz transformations first in the today usual form. In 1904 a paper by Lorentz appeared in which the optics of moving bodies was treated entirely from the standpoint of Lorentz transformations. In 1905 Poincarè and Einstein independently drew the following conclusions from the Lorentz transformations:

- 1. Length contraction and time dilation from the point of view of moving observers*
- 2. Addition theorem of velocities*
- 3. Doppler effect*

Einstein's work also contained the fundamental law of the inertia of energy.

Detailed information about the development of the theory of relativity contains the book "Refined is the Lord" by Abraham Pais.

This book shows that Poincarè presented his work on 5.6.1905 in the Académie des Sciences (Paris) and Einstein published his work on 30.6.1905 in the Annalen der Physik.

The special merit of Poincarè and Einstein for the special relativity theory is the proof that the Lorentz transformations are in accordance with the principle of relativity. Only when it was clear what is meant by simultaneity of events (synchronization of clocks), which take place at different places, the precondition for such a proof was given. Einstein had to face this question during his activity in the patent office in Bern. The reason for this were problems concerning the departure and arrival times of long-distance trains - think of the Orient Express.

*„The same leaves us alone, but it's the contradiction
that makes us productive.“*

Johann Wolfgang von Goethe

